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THE UNIVERSITY OF TEXAS BULLETIN

No. 3706: February 8, 1937

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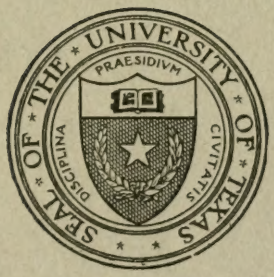
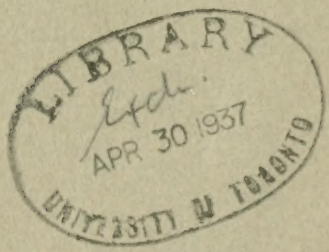
THE TEXAS MATHEMATICS TEACHERS' BULLETIN

Volume XXI





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THE UNIVERSITY OF TEXAS PRESS

THE UNIVERSITY OF TEXAS BULLETIN
No. 3706: February 8, 1937

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Volume XXI

PUBLISHED BY THE UNIVERSITY FOUR TIMES A MONTH AND ENTERED AS SECOND-CLASS MATTER AT THE POST OFFICE AT AUSTIN, TEXAS, UNDER THE ACT OF AUGUST 24, 1912

The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge and the only security which freemen desire.

Mirabeau B. Lamar

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THE TEXAS MATHEMATICS TEACHERS' BULLETIN

Volume XXI

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This bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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"As the drill will not penetrate the granite unless kept to the work hour after hour, so the mind will not penetrate the secrets of mathematics unless held long and vigorously to the work. As the sun's rays burn only when concentrated, so the mind achieves mastery in mathematics, and indeed in every branch of knowledge, only when its possessor hurls all his forces upon it. Mathematics, like all the other sciences, opens its doors to those only who knock long and hard. No more damaging evidence can be adduced to prove the weakness of character than for one to have aversion to mathematics; for whether one wishes so or not, it is nevertheless true; that to have aversion for mathematics means to have aversion to accurate, painstaking, and persistent hard study, and to have aversion to hard study is to fail to secure a liberal education, and thus fail to compete in that fierce and vigorous struggle for the highest and the truest and the best in life which only the strong can hope to secure."—B. F. Finkel. (From *Mathematical Nuts*, by S. I. Jones, page v.)

THE NEED FOR TRAINING TEACHERS OF ARITHMETIC TO BE INTELLIGENT USERS OF NUMBERS INSTEAD OF POOR JUGGLERS OF FIGURES

By

MARY RUTH COOK

North Texas State Teachers College

If the purpose of teaching arithmetic is to enable pupils to think in quantitative situations so as to handle intelligently the quantitative problems they meet in everyday life and will meet in adult life, teachers of arithmetic are faced with the problem of how to teach pupils to be intelligent users of numbers instead of poor, or even clever, jugglers of figures. There is a great deal of criticism today from all walks of life of the contribution which arithmetic, as it is now taught, is making to the child. The college professors point out that their students are unable to handle the very simplest arithmetical situations which they meet in such courses as chemistry, physics, economics, mechanical drawing, shop work, home economics, to say nothing of mathematics. Even more serious indictments are brought by critics in the business world.

Pressey and Moore point out that:

> Inadequate mastery of fundamental terminology is one of the most important reasons for the difficulty encountered by so many persons of all ages and social strata in dealing with anything of a mathematical nature. They have never mastered these relatively simple, fundamental meanings, and their efforts to build on a shaky foundation have been so futile that they have come to regard mathematics as a subject which can be mastered only by those with a genius for figures.[1]

[1] "The Growth of Mathematical Vocabulary from the Third Grade through High School" by L. C. Pressey and W. S. Moore, *School Review*, June, 1932.

In conducting a course on The Teaching of Arithmetic in the Elementary Grades for prospective elementary school teachers, the writer has become more and more convinced that the reason students and average citizens do not possess the fundamental concepts essential to intelligent use of arithmetic is that the average teacher who goes out to instruct children in their arithmetic from grades one through six—those important years when foundations are laid and attitudes formed—cannot read intelligently arithmetical materials. If these teachers are not intelligent users of numbers, is it any wonder that the students they instruct come to look upon arithmetic as a subject to be mastered only by some genius who by a gift of magic can juggle figures?

There can surely be no doubt of the fact that the students who gave the following problems in answer to the directions: "Write a problem that could be used to introduce a class of third grade pupils to the division combination $48 \div 6$. You may think of the combination stated in either of its possible ways, i.e., $48 \div 6$ or $48 \div 8$" were not intelligent enough thinkers in quantitative situations to be able to help children become intelligent users of numbers:

1. A teacher had 48 story books to distribute equally among a group of 48 children. How many did each child get? The next day 2 new members were added to the class. How many books will each child get if the teacher distributed the same number of books among the children that she did the day before?
2. Six boys bought some marbles for 48c. How many marbles did each boy get? If there were 8 boys how many marbles would each boy get?
3. If Bill had 48 plums and 6 bowls and he wanted to put an equal amount in each bowl. How many plums would he put in the 6 bowls?
4. Have the children to make a scrapbook for Mothers' Day and paste pictures and poems about mother. If there are 48 sheets of paper and it is large enough to cut 6 small sheets, how many sheets would you get out of 48?
5. A girl bought some ribbon for 48c. She divided it equally between 8 girls that belonged to her club. How much ribbon did each girl get?

6. If Mary had 48c and she wanted to take out money three different times, how much money would she take out each time?
7. If a man had 6 packages of paper and there were 48 sheets in each package, how much would he have to pay for each package?

The above stated problems were actually written by prospective elementary school teachers.

In an attempt to determine to what extent a representative group of prospective elementary school teachers possessed clear concepts of some of the fundamental arithmetic terms and, further, in an attempt to impress upon these prospective teachers the importance of their having a well developed background in mathematical terminology in order that they might develop real understanding and intelligent use of numbers on the part of their pupils, the writer made the following investigation. A random list of 100 technical words commonly used in arithmetic from grades one through seven was compiled. No attempt was made to check the list with lists previously used in research, nor to check the frequency with which the terms occurred in text books. No attempt was made to distribute the words as to grade levels. A group of thirty college students—one last semester sophomore, fourteen juniors, and fifteen seniors—elementary education majors, were asked to take the test. The group had studied the teaching of reading, writing, and counting numbers, and the four fundamental operations with integers. The following directions were given: "Explain briefly but concisely the meaning of each of the following as they are used in arithmetic. Give a concrete example or illustration also if this will help to make the meaning clear." The test did not call for routine or formal definitions, but rather for an explanation which would reflect clear understanding of the meanings. The responses of this test were classified under five captions—Clearly Understood, Understood but Poorly Expressed, Doubtful, Definitely Missed, and Omitted. The limits of

this article will not permit quoting all of the statements given; however, a representative number will be given.

Explanations that were correct, but were not definitely enough stated to indicate complete comprehension were termed doubtful. For example the following explanations were counted doubtful:

Polygon denotes the shape of a figure with so many sides.
Rod is a linear measure.
Bond is printed paper issued by a company or incorporation.
Formula is mathematical directions.
Meter is a unit of measure.
Liter is a measure used in chemistry.
Volume is the amount that a cylindrical object holds.
Decimal fraction is another way of expressing a part of a whole number.
Acre is a certain amount of land.

The following will illustrate what was characterized "Understood but Poorly Expressed":

Per cent is the ratio of one number to another; as 3 is 50% of 6.
Highest common factor of two or more numbers is finding the biggest factor; as 12, 18, 24—H. C. F. is 6.
Multiply is where you take two numbers and multiply them together like 8 times 9 is 72.
Diameter is the width of the circle.
Denominator is the bottom number of a fraction.
Reciprocal is changing a number upside down as one over two to two over one.

Some of the responses gave evidence of the fact that comprehension of the meaning of a word or expression in some uses does not assure comprehension of its use in mathematics. In most cases these responses were classified as missed. The following are illustrative of this type of response:

Altitude is the distance above sea level.
Function is the position or "job" of anything.
Bond is an amount of money held as security over a person.
Bond is a close connection or tie formed between two things.
Notation is a thing to be noted.

Meter is a device where things can be registered on. Example gas meter.
Volume is a book out of a series.
Volume is the strength of a thing or sound.
Commission is something to be fulfilled.

It was interesting to note the many and widely varying misconceptions of the same expression. For example:

Prime number was explained as:
The essential number.
The small number written just above the concrete number which designates the power of a number.
Numbers 1, 3, 5, 7, 9.
Any number through nine.
The first number. We speak of prime factors and mean the first or smallest factors. For example the prime factors of 8 are 2 times 4.
To note likenesses and differences, for instance *C* prime is a little different from *C* double prime.
Is one tried as in division it is spoken of as a trial divisor.
The first number.

Rectangle was explained as:
A many-sided plane.
A four-sided figure.
A four-sided figure having two sides equal.
A four-sided figure having two opposite sides longer than the other two sides.
A figure that has six sides.
A four-sided figure that has two sides shorter than the other two.
A figure with opposite sides equal.
A angle containing 180° having four sides.
A figure with the opposite sides parallel.
A box-shaped thing, longer one way than the other. Its opposite sides are parallel.
A plane having four sides with opposite sides parallel.
A four-sided figure with two sets of lines with equal length.
A figure with four sides but are not even.
A four-sided figure with two sides just alike.
A figure where the length is longer than the width.

Trapezoid was explained as:
A five-sided solid.
A five-sided figure.
A figure containing five unequal sides.
A plane bounded by four or more sides of unequal length.

A four-sided figure with uneven lines.
A figure with three sides and not equal to each other.
A figure with sides of unequal length.
An eight-sided figure.

Diameter was explained as:

The width of something as the distance between two points.
The distance through the center.
The distance through the center of a triangle from one side to another.
The distance through a number.
The length around something.
The distance from one side to the opposite side.
The size of a circle.

A few of the most absurd explanations were:

Concrete number is a big number.
Concrete number can be broken down evenly.
Ratio is the area of a problem.
Altitude is the height of the atmosphere.
Altitude is an imaginary line drawn perpendicular to the base of a triangle *disecting* the upper angle.
List price is a list of things and their prices.
Lowest common multiple of two or more numbers. The lowest common multiplier is 1.
Abstract number is working a problem with chalk, pencils or chairs, and working out more on paper.
Reciprocal is a relative of a number.
Base of a triangle is the two sides other than the altitude.
Integer is a part of a fraction.
Improper fraction is a fraction not divisible by two.
Subtrahend is a number that is being multiplied by another.
Partial product is paying part of a payment that is due.
Equation is a fractional problem.
Equation. $3a+b$ is an equation.
Divide. If we know the number of pupils in a room we can find the number of boys by the number of girls by knowing to divide the number of girls into the number of pupils in all.
Bond may be a [ ], ( ), or $\{\ \}$.
Tax is a fee or charge for usury.
Least common denominator of two or more fractions is the smallest number you can think of when dividing.
Perpendicular lines are lines which are drawn up and down at the same time are apart at both ends.

Perpendicular lines are lines that meet or are equal to each other.

Millimeter are numbers given on a yard stick or any measuring device. A millimeter may be small or large. It's just according to what's being measured.

Area. The distance around a circle is called its area.

Area. The area of a county or ball is its size, distance around and through.

Mixed numbers are even and odd numbers.

Liter is a term for expressing a group of a certain thing—Pigs!

The following chart gives the list of terms and expressions included in the test, together with the number classified under each caption:

Term or Expression	Clearly Understood	Understood But Poorly Expressed	Doubtful	Definitely Missed	Omitted
1. triangle	20	5	0	5	0
2. lowest common multiple of two or more numbers	1	1	1	18	9
3. power of a number	9	2	1	7	11
4. per cent	5	10	5	9	1
5. root of a number	2	3	7	11	7
6. perimeter	8	0	0	7	15
7. cardinal meaning of numbers	12	5	1	10	2
8. concrete numbers	10	1	4	5	10
9. prime numbers	2	0	0	10	18
10. remainder	24	3	0	2	1
11. numeration	10	1	2	7	10
12. parallel lines	13	7	2	7	1
13. minuend	24	0	0	6	0
14. rectangle	3	1	0	26	0
15. factor of a number	11	6	3	2	8
16. ratio	10	5	5	4	6
17. highest common factor of two or more numbers	10	3	0	8	9
18. trapezoid	6	0	0	12	12
19. higher decade number	17	1	2	9	1
20. ordinal meaning of numbers	9	0	8	10	3
21. altitude	3	4	13	5	5
22. multiply	17	10	0	1	2
23. list price	8	2	3	9	8
24. graph	7	13	1	5	4
25. diameter	6	5	1	14	4
26. interest	10	9	2	4	5
27. digit	7	8	7	5	3
28. abstract number	8	5	4	3	10
29. denominator	1	18	1	4	6
30. variable	2	2	5	3	18
31. reciprocal	0	4	0	6	20
32. trial quotient	10	1	2	5	12

Term or Expression	Clearly Understood	Understood But Poorly Expressed	Doubtful	Definitely Missed	Omitted
33. subtract	26	3	0	0	4
34. base of a triangle	16	5	0	5	4
35. primary number	22	0	1	5	2
36. multiplicand	20	3	0	5	2
37. rote counting	17	0	6	3	4
38. angle	12	5	3	8	2
39. add	21	7	0	1	1
40. percentage	0	2	3	16	9
41. proportion	1	4	5	7	13
42. dividend	26	0	0	3	1
43. addend	23	2	2	1	2
44. quadrilateral	10	1	0	4	15
45. multiple of a number	0	2	0	16	12
46. integer	6	0	3	6	15
47. improper fraction	5	5	0	11	9
48. polygon	8	1	1	11	9
49. circulating or repeating decimal	2	1	0	1	26
50. subtrahend	19	0	1	8	2
51. stock	6	5	6	2	11
52. partial product	7	4	1	2	16
53. rod	4	0	16	6	4
54. equation	5	3	3	11	8
55. promissory note	14	3	0	3	10
56. divide	19	6	1	1	3
57. function	0	1	0	17	12
58. bond	1	1	3	12	13
59. quotient	25	0	0	2	3
60. tax	9	7	3	4	7
61. least common denominator of two or more fractions	9	3	2	13	3
62. perpendicular lines	11	0	4	15	0
63. millimeter	0	0	9	8	13
64. product	27	0	0	2	1
65. kilometer	0	0	12	2	16
66. divisor	29	0	0	0	1
67. cube	3	1	6	11	9
68. rational counting	24	1	0	3	2
69. area	10	5	2	7	6
70. mixed number	19	3	0	4	4
71. inventory	19	2	0	1	8
72. radius	15	2	0	6	7
73. demand note	5	2	2	2	19
74. equilateral	8	1	2	5	14
75. discount	16	4	2	4	4
76. sum	27	1	0	0	2
77. column	20	6	0	1	3
78. formula	6	6	4	8	6
79. constant	6	4	2	2	16
80. circumference	15	0	1	9	5
81. problem	16	3	1	2	8
82. square	3	4	0	20	3
83. notation	5	2	2	4	17
84. proper fraction	8	3	0	6	13
85. meter	1	0	11	7	11

Term or Expression	Clearly Understood	Understood But Poorly Expressed	Doubtful	Definitely Missed	Omitted
86. denominate number	0	0	0	8	22
87. mixed decimal fraction	5	2	0	9	14
88. equiangular	11	0	1	1	17
89. liter	2	1	8	2	17
90. insurance	11	5	2	1	11
91. volume	6	1	2	8	13
92. decimal fraction	0	14	3	3	10
93. acre	5	4	6	4	11
94. commission	11	6	3	2	8
95. numerator	1	18	0	3	8
96. multiplier	21	3	0	0	6
97. common fraction	5	8	2	2	13
98. net price	3	0	4	9	14
99. average	6	10	1	7	6
100. exceed	15	0	1	2	12

The large number showing a clear understanding of the following terms as shown by combining the numbers under "Clearly Understood" and "Understood but Poorly Expressed" was probably due to the fact that these topics had been discussed in the course: remainder (27), minuend (24), multiply (27), subtract (29), primary number (22), multiplicand (23), add (28), dividend (26), addend (25), subtrahend (19), divide (25), quotient (25), product (27), divisor (29), rational counting (25), sum (28), column (26), multiplier (24). To get a better picture of the expressions "Missed" the numbers under the caption "Omitted" should probably have been added to the ones under "Definitely Missed"; for, no doubt, had the students clearly understood these terms they would not have been omitted. After combining these two columns it was significant to note that *denominate number* was not satisfactorily explained by any of the thirty students. *Common multiple* was missed by 27 out of the 30 students, *perimeter* by 22, *prime number* by 28, *rectangle* by 26, *trapezoid* by 24, *reciprocal* by 26, *trial quotient* by 17, *percentage* by 25, *quadrilateral* by 19, *integer* by 21, *improper fraction* and *polygon* each by 20, *bond* by 25, and *demand note* by 21.

The investigation is too limited to prove anything, but it tends to indicate that:

1. Reading skills and the vocabulary of arithmetic—comprehension of arithmetical materials—should be one of

the essentials of arithmetic instruction. While drill on computation is necessary, arithmetic in the primary and intermediate grades should not be treated as a pure drill subject.

2. The comprehension of arithmetic materials should be developed systematically throughout the child's arithmetic experiences by paralleling the training in computation and in comprehension.
3. There is need for more scientific investigation concerning what constitutes the fundamental technical arithmetic concept background for intelligent use of numbers.
4. Training in the special reading skills required to read arithmetic materials with real comprehension should start in the primary grades and be carefully and gradually developed through the years.
5. Prospective teachers and, without doubt, many teachers in the field do not possess the basic concepts of arithmetic fundamental to intelligent reading of arithmetic materials.
6. Without being able to respond intelligently to quantitative terms himself, the teacher cannot train pupils to be intelligent users of numbers.
7. No teacher should be permitted to teach arithmetic until he himself can think and speak the language of arithmetic.
8. Any institution certificating primary and elementary school teachers should provide a well planned and carefully taught professionalized course in arithmetic as a requisite for certification.
9. The need today seems to be not for less arithmetic in our curriculum, but for more intelligent teaching of arithmetic.
10. It is only through having teachers who are understanding, appreciative, and respectful users of numbers that students can be taught to be intelligent users of numbers rather than poor jugglers of figures.

ON FINDING THE SQUARE ROOT OF A NUMBER

By

H. J. ETTLINGER

The University of Texas

The method of finding the square root of a number here presented has many points of interest. The writer lays no claim to originality with respect to it. Professor M. B. Porter of the Department of Pure Mathematics at The University of Texas called my attention to its possibilities. This present method makes use only of long division. On the other hand, the method commonly taught is based on the binomial theorem and is merely a "rule of thumb" to most beginners.

We will illustrate the method by finding the square root of 2 to four figures. By definition, this is a number which gives the product 2 when multiplied by itself. We shall assume that there is such a number and call it r. Since $1^2 = 1$ and $2^2 = 4$, we have

$$1 < r < 2.$$

We will split the difference between 1 and 2 and now divide 2 by 1.5.

$$\begin{array}{r} 1.5)2.0\ (1.3 \\ 1.5 \\ \hline 50 \\ 45 \\ \hline 5 \end{array}$$

We conclude that 1.5 is too large and 1.3 is too small for our answer, that is,

$$1.3 < r < 1.5.$$

We therefore split the difference again and divide 2 by 1.4.

$$
\begin{array}{r}
1.4)\,2.00\ (1.42 \\
1.4 \\
\hline
60 \\
56 \\
\hline
40 \\
28 \\
\hline
12
\end{array}
$$

We now see that 1.4 is too small and 1.42 is too large, or

$$1.40 < r < 1.42.$$

Proceeding as before, we divide 2 by 1.41.

$$
\begin{array}{r}
1.41)\,2.0000\ (1.418 \\
1.41 \\
\hline
590 \\
564 \\
\hline
260 \\
141 \\
\hline
1190 \\
1128 \\
\hline
62
\end{array}
$$

This yields

$$1.41 < r < 1.418.$$

Divide 2 by 1.414:

$$
\begin{array}{r}
1.414)\,2.000000\ (1.4144 \\
1414 \\
\hline
5860 \\
5656 \\
\hline
2040 \\
1414 \\
\hline
6260 \\
5656 \\
\hline
6040
\end{array}
$$

Our final statement for the given problem is that r is between 1.414 and 1.4144 and hence the first four figures of r

are 1.414. We call this result "the square root of 2" written to four significant figures or to three decimal places.

This method is useful and easily presented. It will give the result to any desired number of decimal places. It is self-correcting, in that an error at any stage in the computation will not affect the final result if the process is carried several steps beyond the error. It does not rely on an accurate initial estimate. In the case where the desired square root is an irrational number, it sets forth a definition of such a number as the limit of a sequence of rational numbers which can be written as ascending in magnitude constantly, or decreasing in magnitude constantly. There is also exhibited an estimate of the error in the approximation to which we have calculated the square root.

ELEMENTARY MATHEMATICS CURRICULUM

By

MINNIE S. BEHRENS

Sam Houston State Teachers College, Huntsville, Texas

The teacher as a builder of the elementary curriculum must give the first consideration to two schools of thought. The formal school gives the emphasis to logically arranged subject matter and systematized drill. When we consider the subjects of the elementary field, mathematics is one in which the formal procedure plays an important part. Three hundred and ninety simple facts that serve as a foundation for all future work have to be memorized. All of the upper decade facts plus those in fractions, decimals, and measurements must be added to these.

The committee has submitted a suggestive set-up of subject matter for grades one to six, inclusive. Every teacher who is interested should try out this material and offer suggestions for additions or omissions with reference to the topics that should be taught in elementary mathematics.

At the same time we as mathematics teachers need to give some consideration to the informal or the progressive school. This school believes in introducing activities which provide for purposeful experiences. For example, before a child learns rote counting he is given an opportunity to count objects such as blocks, beads, and the number of boys and girls in class. In the teaching of addition and subtraction activities like playing store or planning a party are introduced in order to develop concepts or understandings of number ideas; in other words, to make numbers meaningful. It is absurd to rely wholly on activities to teach number facts or to develop all phases of mathematics with each social science unit. We could introduce more mathematics into many of our activities, some of which should emphasize mathematics chiefly.

This means then that experiences to give children the understanding by means of doing or working with num-

bers must be provided. After they understand, systematized drill should follow.

In order to make this part of the course of study useful, every teacher, after developing a few activities during the coming year, should submit them to the committee. If the activity units which have been suggested are not the best for your locality, make additions and suggestions of others which you have found to be profitable; if the units which are mentioned in the tentative course of study can be used, please mention the fact.

The child's attitude when school work is too formal should be watched lest he feel that he is rendering the teacher a favor when he studies his assignment. The progressive point of view might go too far in socializing everything or permitting too much freedom. We must encourage freedom with responsibility.

The new course of study should outline subject matter with teaching procedures and suggestive activities which will help children to work with numbers in a meaningful way or encourage knowing plus doing.

The other fundamental step in curriculum building is to give careful consideration to the selection and placement of subject matter. If the Mathematics Course of Study is to be worth while, problems typical of East, West, North, South, and Central Texas must be included in the final course of study. Teachers in the various parts of Texas should make contributions.

When we consider placement of subject matter the following aspect needs to be given careful consideration: the time to begin systematic drill in arithmetic. The emphasis in the first grade should be on the development of the child's reading ability. Children in the second and third grades will do better work in numbers when reading becomes the core and the development of number concepts and numerical experiences becomes a subordinate part of the instruction in the first grade. Many studies show that some topics in the school curriculum are commonly taught at the wrong time, a procedure which dooms a child to

failure. We must give more consideration to the child's mental growth. The child must first understand before he can use numbers effectively.

Certain phases of subject matter are harder than others. For example, to the children numbers with digits 6, 7, 8, and 9 are more difficult than those with digits 1, 2, 3, 4, 5. Naturally all processes in addition, subtraction, multiplication, and division will be harder when the larger digits are used.

Long division is one of the hardest topics in elementary mathematics. The child can master division with greater ease in the fifth grade than in the fourth.[1, 2] Authorities like Washburn have concluded that long division is one of the most difficult topics in arithmetic and "is definitely a sixth grade subject."[3]

Many children below average intelligence find long division impossible to master. The new arithmetic curriculum tends toward shifting long division to the fifth grade and many of the most difficult processes to an even more advanced grade level.

Arithmetic needs to be individualized more than most other subjects because each new process depends on the mastery of the preceding ones. There are three essential steps in the technique of individualization:

1. The development of the number concept before drill is given on each of the fundamental processes. This implies that teachers must have the ability to diagnose the ways or methods which the child uses in learning to add, subtract, multiply, and divide. Children derive answers in all of the fundamental processes in many ways. In addition and subtraction they use various associative processes similar to the following:

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[1]Osburn, W. J., *Corrective Arithmetic*, Volume I, p. 92.

[2]Washburn, C., "Grade Placement of Topics," *Twenty-ninth Yearbook*, National Society for the Study of Education, pp. 656ff.

[3]Washburn, C., *Adjusting the School to the Child*, p. 45.

1. Counting forward in order to add: $7 + 3$; (7), 8, 9, 10; in subtraction, counting backward: $9 - 2$; (9), 8, 7.
2. Counting by 5's or 10's: $5 + 7$; $5 + 5 = 10$, $10 + 2 = 12$.
3. Double combination serves as a basis: $6 + 7$; $6 + 6 = 12$, $6 + 6 + 1 = 13$; in subtraction: $13 - 7$; $7 + 7 = 14$, $14 - 1 = 13$.

At first, poor pupils use and need crutches. If they are not taught better, these children when required to master the process of long division have an accumulation of all the slow and laborious methods of addition, subtraction, and multiplication to contend with, besides those that they continue to invent in order to master division. Crutches too long continued become a hindrance.

In learning subtraction, many children use the additive method if the teacher has placed emphasis on the learning of the addition facts first. On the contrary, many first graders will make up original problems and use the subtraction method first. This may be because in their experiences giving away meant getting for themselves the most cherries.

In naught combinations, superior pupils have very little difficulty in adding, subtracting, and multiplying. The average and slow pupils need intensive drill because the bonds in adding, $3 + 0 = 3$, and subtracting, $3 - 0 = 3$, have been so firmly established that multiplying by naught, $3 \times 0 = 0$, becomes one of the most difficult processes.

In the lower grades we lead children to believe that the product of two numbers multiplied together is always larger than the multiplier or multiplicand. Little do we realize that in the upper grades when the child is introduced to fractions and decimals this idea has to be changed. The same mistake is made in teaching division. Since division is the reverse of multiplication, the child concludes that dividing two numbers will give for the answer a number smaller than either the dividend or the divisor, or that the divisor is smaller than the dividend. If a whole number is divided by a fraction or decimal, the quotient will be larger than either the dividend or the divisor. In the lower

grades teachers should introduce principles which give the child the right numerical concepts,—concepts that serve as foundations for advanced mathematics.

2. If children are to appreciate numbers, they must use them in meaningful situations. Opportunities should be provided to allow children to bring problems from their homes and shops. This necessitates the building of a vocabulary. If the child is able to understand a numerical life situation, he must also be able to express it. In the primary grades chart lessons introducing numerical words and concepts should be developed and number stories read. Reading must precede arithmetic because the making of original problems and the solving of verbal problems must keep pace with the development of skills in mastering fundamental processes.

3. The learning of the basic facts in arithmetic by means of a teaching and testing program must provide for individual differences. A course of study in arithmetic should make some provision for diagnostic and remedial work. We have ample proof that with five minutes of daily individualized drill the superior child can master the one hundred addition facts in two to three weeks; the average pupils need six to nineteen weeks; the inferior pupil may master them in six months; and the very inferior child may never attain complete mastery. In many situations superior children are penalized by repeating what they know. The poor pupils are given most of the attention by excess drill, when in reality they need numbers related to practical situations. The teacher's program must provide for initial testing in order to note which pupils need more advanced work and which need more help. Testing should be followed by teaching, and teaching by retesting. When the processes have been mastered and a new topic is introduced, a maintenance test should be given in order to check whether or not former skills have been retained.

It is important to check on pupil progress. Since every teacher and child wants to see progress, we want to improve our previous methods of checking it. During the

past the emphasis has been on group progress. If we want to develop initiative and confidence in the child, let him race with himself and check himself in terms of his past rather than in terms of the group. If a child is treated as mediocre, he will remain mediocre. Let us start where the child is and help him to achieve what he can. Give him faith and courage while he is trying to achieve.

To summarize, teachers who are interested in developing the new course of study in elementary mathematics should do as follows:

1. Present and encourage the social and purely mathematical activities which will develop the child's number concepts and establish the fundamental principles of mathematics.
2. Give careful consideration to the proper selection of subject matter which is typical of the numerical experiences of today.
3. Help in the proper grade placement of subject matter in terms of the child's mental growth and development.
4. Present a program that will provide and suggest for children of all levels the best teaching procedures; the minimum essentials for pupils of the lowest as well as for other grade levels, and an enriched program for superior children.

The State of Texas needs the help of every teacher. Send in your suggestions whether they are critical or complimentary of what has been planned. This course must prove beneficial to all children of Texas in order that the child may learn to think and judge for himself.

A GLIMPSE OF ACTUARIAL MATHEMATICS

By

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1. *Introduction, Net premiums, reserves, loading*

The mathematics needed by an actuary of a large insurance company is too extensive even to summarize properly in a short paper. Of the actuary's three major problems, (1) the *net premium*, (2) the *reserve*, and (3) the *loading*, only the first will be considered in this very brief sketch. A *net premium* is a charge for a promised benefit to an insured person or prospective annuitant such that the total net premiums collected will exactly discharge the obligations—assuming usually some standard or legalized mortality table and rate of interest—without considering expenses or profits. In the actual charge or *gross premium*, there is included a *loading* for expenses—and profits, if any. By the load may be meant (1) the actual excess of the gross premium over the above net premium, or (2) this excess diminished by the dividend given in participating insurance, or (3) the excess of the gross premium over the net premium determined from a company's own computed mortality and interest rates, as distinguished from the rates prescribed by law for the *reserves* which companies are obliged to maintain to protect their contracts.

2. *Compound interest, perpetuities, and annuities certain*

To understand the mathematics of life insurance, some knowledge of compound interest is needed. If \$100 is at 4% interest for a year, the amount is \$104. This \$104 then becomes principal; and the amount at the end of the second year is $\$108.16 = \$100(1.04)^2$. In conformity with this, if r is an interest rate for a specified period, then the amount S of P dollars at rate r for t periods is *defined* to be

$$S = P(1 + r)^t. \tag{1}$$

This is indeed a definition, because we permit t to be any real positive number. From this, there is obtained the important equation

$$P = S(1 + r)^{-t}, \tag{2}$$

from which we compute the value P of the sum S payable t periods later.

As used in the headings of monetary tables, the letter i is commonly used, like r above, to refer to any period. But in general theory this i often refers specifically to the *interest rate* for *one year*. Also d is used for the *discount rate* for *one year;* or, as it may be called, the rate of interest in advance. Thus, \$1 due in one year is now worth $1 - d$; and it is convenient to write

$$v = 1 - d. \tag{3}$$

It is easily proved that

$$\begin{aligned} d = i\ (1 - i) &= i - i^2 + i^3 - i^4 + \ . \ . \ . \\ &= i - i^2, \text{ approximately.} \end{aligned} \tag{4}$$

For example, with a 4% interest rate, $i = .04$, $d = .04 - .0016 = .0384$, approximately, the error being less than .0001.

The i as an annal rate is often called an *effective* rate, to distinguish it from *nominal* interest rates. Thus, if 2% is added every half-year, we say that money is at a *nominal* 4%, convertible semiannually; and we write $j_2 = .04$. In general, with j_m, there are m conversions of interest; and when the symbol f_m is used as a *nominal discount* rate, it is understood that there are m conversions a year. The following formulas hold:

$$1 + i = \left(1 + \frac{j_m}{m}\right)^m;\ 1 - d = \left(1 - \frac{f_m}{m}\right)^m. \tag{5}$$

If now, with i fixed, m is allowed to increase indefinitely, both j_m and f_m approach a common limit called the *force of interest*, δ.

(6) $\lim j_m = \lim f_m = \delta$; $\delta = \log_e (1 + i)$; $e = 2.71828 \ldots$.

This δ is just about half-way between d and i. Indeed, with $i = .04$, $\delta = (d + i)/2 = .0392$, correct to four decimal places. The following inequalities obtain:

$$(7) \qquad d < f_m < \delta < j_m < i.$$

Since d differs from i by less than i^2 in the usual case of $i < 1$, the above rates are in general rather close together; but they should not be confused in monetary or actuarial problems.

From the definition of an interest *rate* as a rate *per dollar* invested, about the simplest deduction that can be made in the mathematics of finance is that if \$1 will yield as interest i per year, forever, then $1/i$ will yield \$1 per year forever. A *perpetuity* of \$1 per year has then the value $1/i$. Likewise, for a *perpetuity due*, paying \$1 per year in advance, forever, the present value is $1/d$. In symbols:

$$(8) \quad a_{\overline{\infty}|} = \frac{1}{i}, \; a^{(m)}_{\overline{\infty}|} = \frac{1}{j_m}, \; \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}, \; a^{(m)}_{\overline{\infty}|} = \frac{1}{f_m}, \; a_{\overline{\infty}|} = \frac{1}{d}.$$

This states that if \$1 per year is paid in installments of $1/m$ each, at the *beginning* of each interval of $(1/m)^{th}$ year, forever, the present value is $1/f_m$. If the dollar is paid "*continuously*"—say a penny every three or four days—the present value is $1/\delta$.

An important variation occurs when there is a *deferment* for n years, the first payment being made n years later than otherwise contemplated. To adjust for this, the multiplier $(1 + i)^{-n}$ in (2) is used; more briefly written v^n, by (3). Thus, the ordinary perpetuity, if deferred n years, has a present value of v^n/i.

Now, an *annuity* of \$1 payable at the *end* of each year for n years may be thought of as the difference between a perpetuity and a deferred perpetuity. Let $a_{\overline{n}|}$ be its present value. Then

$$(9) \qquad a_{\overline{n}|} = \frac{1}{i} - \left[\frac{1}{i}\right] v^n = \frac{1 - v^n}{i}.$$

Likewise, for an *annuity due*, with first payment immediate,

$$(10) \qquad a_{\overline{n}|} = \frac{1}{d} - \left|\frac{1}{d}\right| v^n = \frac{1 - v^n}{d};$$

or, as it may be written:

$$(11) \qquad \frac{1}{d} = a_{\overline{n}} + \left|\frac{1}{d}\right| v^n.$$

This expresses the fact that the value of a perpetuity due $1/d$ may be split up into two parts—the first part $a_{\overline{n}}$ covering the value of the first n payments, and the second part $v^n\ d$ the value of of all the remaining payments. An analogue of this will be taken up presently as one of the most important formulas in actuarial mathematics.

3. *Annuities and insurance on a single life*

For annuity and insurance problems, some mortality table is assumed. In the United States, the table in most common use is the American Experience Table. The first and last entries in this table are as follows:

Age	Living at age x	Dying at age x
x	l_x	d_x
10	100,000	749
11	99,251	746
12	98,505	743
. . .	. . .	. . .
92	216	137
93	79	58
94	21	18
95	3	3

The value to a man of age 92 of \$1 payable to him at the end of each year that he survives is found as follows, on a 3–1/2% interest basis:

$$(12) \qquad a_{92} = (v\, l_{93} + v^2 l_{94} + v^3 l_{95})\ l_{92}^{-1}$$
$$= (79v + 21v^2 + 3v^3)/216 = 0.4567,$$

where $v = 1/(1.035)$. This is the pro-rata for each of 216 men alive at age 92 of the present values of the payments of \$1 each to the 79 men expected to survive one year, to the 21 men surviving two years, and to the 3 men surviving 3 years. We also write

$$a_{92} = 1 + a_{92} = 1.4567$$

with this *life-annuity due* providing an extra dollar at once. It is convenient to write also:

$$(13) \quad a_{92} = v_1 p_{92} + v^2{}_2 p_{92} + v^3{}_3 p_{92} = v(79/216) + v^2(21/216) + v^3(3/216).$$

Thus, since it is estimated that of 216 men alive at age 92, only 3 will reach age 95, it seems proper to say that a man of age 92 has only 3 chances in 216 or one chance in 72 of living the 3 years. His chance or probability of getting the third dollar is then 3/216 or 1/72. Likewise, 79/216 and 21/216 are the probabilities of his living one or two years, respectively.

In a similar manner, we find the present value to a man of age 92 or \$1 payable at the end of the year of his death as follows:

$$(14) \quad A_{92} = (vd_{92} + v^2 d_{93} + v^3 d_{94} + v^4 d_{95})/l_{92} = 0.95074.$$

Or we may write

$$(15) \quad A_{92} = v(137/216) + v^2(58/216) + v^3(18/216) + v^4(3/216),$$

where the parentheses are the respective probabilities that a man of age 92 will die in the first, second, third, or fourth year.

To simplify the computations, the following *commutation* functions are obtained:

$$(16) \quad \begin{array}{ll} D_x = v^x l_x & C_x = v^{x+1} d_x \\ N_x = D_x + D_{x+1} + D_{x+2} + \ldots & M_x = C_x + C_{x+1} + C_{x+2} + \ldots \\ S_x = N_x + N_{x+1} + D_{x+2} + \ldots & R_x = M_x + M_{x+1} + M_{x+2} + \ldots \end{array}$$

The summations extend to the end of the table. The above N_x and S_x represent American notation.

The $a_{\overline{n}|}$ and $A_{\overline{n}|}$ just considered may be computed thus:

$$a_{\overline{n}|}=N_{\overline{n}|}\ D_{\overline{n}|};\ A_{\overline{n}|}=M_{\overline{n}|}\ D_{\overline{n}|}.$$

With the aid of R_x and S_x, problems in increasing or decreasing annuities or insurance can be solved,—also insurance involving the return of certain premiums.

4. *The continuance and the discontinuance of a status*

In general, annuity and insurance problems are complementary to each other. Thus, as an analogue to (11), we may now write:

$$\frac{1}{d}=a_x+\frac{1}{d}A_x. \tag{17}$$

Here, the a_x provides \$1 per year in advance as long as a certain status—the annuitant originally of age x is alive—persists. That is, a_x is the value of such a benefit to a man now of age x. But A_x is the present value to a man of age x of \$1 payable at the end of the year of his death. Then $A_x\ d$ paid now would provide a *perpetuity due* of \$1 per year forever, beginning at that time. The 1 d on the left side of (17) provides a perpetuity-due, but beginning at once. Thus equation (17) splits the perpetuity-due with present value 1 d into two constituents, the first representing continuance of a status, and the second its discontinuance.

In the *general* problems of *insurance*, *annuities*, *pensions*, and *compensations*, we are interested in *groups* of individuals, such as the *members* of a *family*. A *complex status* may involve *life*, *health*, *occupation*, or *marital condition* for several individuals. To bring out the generality of equation (17), we may now use the symbol σ to signify a status, and solving for $A\sigma$, write

$$A\sigma=1-d\ a\sigma. \tag{18}$$

Here $a\sigma$ provides a dollar a year in advance as long as a specified status persists; $A\sigma$ provides a dollar at the end of the year that the status is broken. If we divide a year into

m equal parts, then $a^{(m)}_{\sigma}$ will provide—by definition—$1/m$ in advance throughout continuance of the status; and $A^{(m)}_{\sigma}$ will provide \$1 at the end of the fractional part of the year that the change occurs. Here we have

$$A^{(m)}_{\sigma} = 1 - f_m a^{(m)}_{\sigma}. \tag{19}$$

Letting m increase indefinitely, we get

$$A\sigma = 1 - \delta\, a\sigma. \tag{20}$$

In general, insurance and annuity problems have their mates. We may solve first the problem that is simplest—usually the annuity problem—and go to (18), (19) or (20) for the solution of the other problem. Indeed, *conversion* tables are constructed to expedite this process.

5. *The symmetric status*

Insurance for a group is sometimes conditioned by the occurrence of deaths in some specified order. The status in this case is not symmetrical. Or, again, if in workmen's compensation, a widow loses her annuity upon remarriage, while a child loses it only through death, the status is not symmetrical. In a *symmetric* status, the *number* of individuals affected in some particular way determines the continuance on discontinuance of the status, without reference to which individuals are affected. Only that case will be treated here.

To this case some general theorems of probability theory may be applied. For s independent events, let $p_1, p_2, \ldots, p_s$ be the respective probabilities of occurrence. Form the symmetric functions:

$$\begin{aligned} Z_1 &= p_1 + p_2 + \ldots + p_s \\ Z_2 &= p_1\, p_2 + p_1\, p_s + \ldots + p_{s-1}\, p_s \\ Z_s &= p_1\, p_2 \ldots p_s. \end{aligned} \tag{21}$$

Here Z_r is the sum of all possible products that can be formed by using as factors exactly r of the s given

probabilities. In actuarial literature, this r is usually printed as a midscript, instead of as a subscript. These Z-functions are to be combined with the binomial coefficients

$$(22) \qquad {}_tC_r = \frac{t!}{r!(t-r)!}; \; t! = 1 \cdot 2 \cdot 3 \cdots (t-1) \cdot t.$$

Indeed, let P_r denote the probability that of the s specified events *at least* r will occur. It can then be proved that

$$(23) \qquad P_r = Z_r - {}_rC_1 \, Z_{r+1} + {}_{r+1}C_2 Z_{r+2} - \ldots + (-1)^{s-r} {}_{s-1}C_{s-r} \, Z_s.$$

Here the terms on the right are alternately positive and negative, and the subscripts increase by unity as we go from one term to the next. If in (23) each pre-subscript of C is increased by unity, the expression gives the probability for the occurrence of *exactly* r of the s events. The probability that *at most* t events will occur is $1 - P_{t+1}$; that *at least* r *and at most* t will occur is $P_r - P_{t+1}$. In applying these theorems to annuities payable under the continuance of a symmetric status, the p's refer respectively to the s individuals in a group; and are used in turn for 1, 2, 3, . . . years, to the end of the table. The discounting factors $v, v^2, v^3, \ldots$ are introduced; and relations between probabilities go over immediately into similar relations between annuities or insurances. The basic *life* annuities are (1) those on single lives, $a_x, a_y, \ldots$; and (2) those on *joint lives*, $a_{xy}, a_{xyz}, \ldots$, where the status is broken when any one of the group under consideration dies. In terms of these, all other symmetric life annuities can be expressed. For example:

$$(24) \qquad a_{\overline{xy}} = a_x + a_y - a_{xy}.$$

This gives the present value of one dollar per year as long as *either* of the two lives persists. Again,

$$(25) \qquad a_{\overline{xyz}}^{2} = a_{xy} + a_{xz} + a_{yz} - 2\,a_{xyz}.$$

This provides a dollar per year as long as *at least two* of the three survive.

If in the foregoing two equations, a is replaced by A throughout, the equations remain valid, giving in each case the present value of \$1 payable at the end of the year in which the *second death* occurs.

6. *The force of mortality, and the Makeham Law of survival*

An immense simplification of certain annuity and insurance problems can be effected if the mortality table used conforms to what is known as *Makeham's Law*. For a brief introduction to this law, let us go back to q_x, the probability that a man of age x will die within a year, and write it:

$$(26) \qquad q_x = \frac{d_x}{l_x} = -\frac{1}{l_x}\frac{l_{x+1} - l_x}{1} = -\frac{1}{l_x}\frac{\triangle l_x}{\triangle x}, \quad \triangle x = 1.$$

As originally exhibited, this l_x takes on certain integral values when the x takes on the integral values, 0, 1, 2 . . ., to the end of the table. But it is convenient to think of l_x as a continuous function

$$(27) \qquad l_x = f(x),$$

such as would appear in calculus, with a derivative, $d\,f(x)/d\,x = d\,l_x/d\,x$. The *force of mortality* μ_x is then defined as

$$(28) \qquad \mu_x = -\frac{1}{l_x}\frac{dl_x}{dx},$$

giving what may be called the *instantaneous rate of dying*—whereas q_x in (26) gives the *annual* rate. The theoretical difficulties in visualizing μ_x are no greater than those involved in defining or describing density in mechanics.

In the differential form, the Makeham Law asserts that the μ_x in (28) is of the form

$$(29) \qquad \mu_x = A + B\,c^x,$$

where A, B, and c are constants. It has been found that the American Experience Table and a number of other

important tables conform very closely to the Makeham Law,—at least at adult ages.

7. *The average or equated age*

Consider now an annuity payable as long as *any one* of several individuals—say, of five individuals—is alive. The value of this is expressible in terms of *simple* annuities and *joint* annuities on two, three, four, and five lives. But if a mortality table extends to age 100, it would require nearly a hundred million entries to give the present values of all the joint annuities on five lives of various ages. For six lives, over a billion such present values would be needed.

With a *Makehamized* table, however, it has been shown that for five ages u, v, x, y, z, there can be *substituted* a single age w such that a joint annuity on five men all of age w will have the same value as a joint annuity at the given ages, provided that either of the two following equivalent relations is satisfied:

(30) $\mu_w = (\mu_u + \mu_v + \mu_x + \mu_y + \mu_z)\ 5;\ c^w = (c^u + c^v + c^x + c^y + c^z)\ 5.$

This shows that the *equated age* w is an *exponential mean* of the given ages. The same rule applies whatever the number of lives considered: the force of mortality at the *equated* age is the arithmetic mean or *common average* of the forces of mortality at the given ages. It should be noted that the equated age *does not depend* on the *interest rate* used in the computations.

For problems considered here, then, it is only necessary to have tables constructed for joint annuities with ages in a group all equal, computed at suitable interest rates.

8. *Deferment and fractional payment*

By reference to (2) and (3) it will be seen that the discounting factor v^n, as used in the mathematics of finance for *annuities certain*, accomplishes a deferment through n years. Let now ${}_np_\sigma$ be the probability that a status σ will persist for n years. Then in actuarial mathematics dealing

with *annuities contingent*, the corresponding function to accomplish the deferment is

$$(31) \qquad {}_nE\sigma = v^n {}_np\sigma,$$

often designated as a *pure endowment*. It is the value at the beginning of the specified status of $1 payable at the end of n years on condition that the status persists. It may well be called a *deferment factor*.

In (18), (19), and (20) are given three form of life annuities, depending upon the interval of time adopted as a unit, the two latter involving *fractional payment*. As approximations, it is customary to take

$$(32) \qquad \mathrm{a}\overset{(m)}{}_\sigma = \mathrm{a}\sigma - (m-1)/2m; \ \bar{a}\sigma = \mathrm{a}\sigma - 1/2.$$

If any deferment is contemplated, (31) should be used *before* (32), also before (18), (19), or (20). Formulas giving closer approximations than (32) are possible, as found from the *calculus of finite differences*.

9. *Conclusion*

The reader will observe that this rapid sketch leaves many problems unsolved; and although the approach to certain problems has been indicated, the mathematical details have been hardly touched. Moreover, mention has not been made of various statistical problems such as the determination of progressive changes in mortality rates, or the fluctuation of interest rates.

However, perhaps the conclusion may be properly drawn that the formulation of financial contracts based on various contingencies incident to an aggregate of human lives calls for a high degree of mathematical skill as well as sound judgment and intimate knowledge of the chief characteristics, physical and psychical, of the group of individuals considered.

THE EFFECTIVE USE OF DRILL IN ARITHMETIC

By

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Although investigations show that arithmetic is one of the most difficult subjects in the elementary school, it must be admitted that it is hard only when pupils are attempting processes containing facts or skills that they have not yet mastered. Statistics show that pupils in the late elementary grades have only about 40 per cent mastery of some of the simpler tasks.[1] There is need, therefore, for the use of diagnostic methods to determine specific weaknesses and the causes of them. Pupils cannot make satisfactory progress, nor use their time economically, until the fundamental skills and processes are mastered in their proper sequence.

Hence, it should be the aim of every teacher of elementary arithmetic to develop a method of diagnosis of pupil difficulties in the fundamental processes of arithmetic, to use a special program for remedial treatment that will operate effectively, and to adopt drill and practice material as a part of the classroom procedure that will stress 100 per cent mastery of every skill.

Arithmetic is unfortunate in the language which has come to be used to describe the processes by which its subject matter is to be learned and taught. Such terms as "number facts," "skills," "100% accuracy," and "drills," have a significance which is seldom sufficiently recognized. For instance, the term "drill" is loosely used in discussions relating to arithmetic instruction. At times "drill" is used to mean a theory of instruction which makes repetition on the part of the pupil the essential of learning.

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[1] Schorling-Clark-Potter Arithmetic Test. *Manual of Directions*, p. 17.

Again, it refers only to the practice and maintenance exercises which follow initial instructions. From the standpoint of the "meaning" theory of arithmetic, drill is recommended when ideas and processes already understood are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse.

Since the ultimate purpose of arithmetic instruction is the development of the ability to *think* in quantitative situations and because the "meaning" theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes, children must be taught to analyze real or described quantitative situations, to isolate, to treat adequately the arithmetical elements therein, and to make whatever adjustments are required by their solutions. Thus true arithmetical learning is seen to be a matter of growth which needs to be carefully checked, controlled, and guided at every stage of development. If the teacher is to check, control, and direct growth, she must do so in terms of the child's method of thinking. Therefore it is expedient that the teacher know the individual weaknesses within the group. Such a diagnostic study will usually make it clear to the teacher that numerous factors interfere with the normal progress of pupils in mastering arithmetic processes. Remedial instruction is concerned with the removal of such factors and the substitution of efficient methods of work for wasteful procedures that pupils have acquired so that these pupils may be kept working all the time at maximum capacity.

The hope for improvement in any remedial program lies in the proper use of drill. We shall consider that drills, like other teaching devices, have only one main purpose, which is to increase the pupil's understanding and facility in the use of desirable facts, skills and information. To get the most out of the time and the energy expended in exercise we must construct or select the type of drill which will serve our purpose best. There are two types of drill

organization now in use, *isolated* and *mixed*.[2] In the isolated type intensive exercise of a few skills is emphasized. For years, authors of arithmetic textbooks have advocated, by their textbook production, the use of isolated drill for maintenance and review purposes.

The second type of drill is the *mixed* drill organization, in which many skills and processes are included in one drill exercise. This type of drill is rapidly increasing in favor and is displacing the isolated drill for some purposes.

The chief place which isolated drill should occupy in the learning process is in connection with the first teaching and learning of new facts, skills and information. One effective way to bring about an impression which is deep, clear-cut and lasting is by intensive, interesting, and concentrated exercise just following first teaching and learning. The trend is toward the use of the isolated drill in connection with first teaching. The "stamping in" process can hardly be accomplished without it. Much of the isolated drill should be anticipatory in nature. For example, preceding multiplication a considerable amount of drill in higher-decade addition would be decidedly effective provided it included the facts to be used in the multiplication exercise.

The best type of drill organization for maintenance purposes is the mixed drill. When the same examples that have been used in making the isolated type series of drills are organized in mixed fashion almost every one of the processes dealt with receives some practice every week. When a good mixed-drill schedule is put in operation following good first teaching in which is included valid isolated drill, the learning curve rises gradually but regularly. In such a program every law of learning—use, disuse, frequency, recency, and effect—is adequately considered, and the fact that each example is different from the next in the mixed drill tends to overcome any tendency toward boredom.

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[2]Repp, Austin C., *Types of Drill in Arithmetic. The Tenth Yearbook of the National Council of Teachers of Mathematics.*

The isolated drill, however, has a second important function. Following the discovery of weakness in any particular phase of a gross skill through the use of diagnostic tests, mixed drills, or other means, isolated drills will prove valuable, for here we have a condition similar to a first learning situation. In some respects it is more difficult than a first teaching situation because wrong response patterns must be broken down before the correct response can be established. Reteaching and eliminating the difficulties is the first step in corrective work, for if pupils do not understand the technique of certain operations, practice exercises containing these operations will be of little value to them. If lack of skill with basic facts causes inaccuracies, drill on these facts will be satisfactory.

Certain factors need to be carefully checked in selecting or constructing drill exercises. Difficulty and cruciality of the facts and skills included in the practice exercise need to be studied. Too often easy processes and combinations are exercised frequently, while the difficult combinations and processes are slighted. Also, those facts and processes which are needed in adult life should be included frequently in drills, and unimportant processes be neglected or even ignored. Again, economical teaching favors the use of self-corrective drill and practice material. Individual differences which exist in the group should be recognized and the drill so organized that each pupil is allowed to work at maximum capacity. Furthermore, a drill, to be effective, should have diagnostic and remedial possibilities. A modified mixed drill in which four or five processes are exercised four or five times in a twenty-minute drill period should be used if best diagnostic and remedial results are to be attained.

Many teachers rightly contend that the arithmetic courses for elementary school children include so many skills that there is little time to teach for understanding. This situation could be overcome partly if more emphasis in our drill and teaching were placed upon relationships.

Our number system contains relationships and connections which, if mastered, should enable children to make progress much more readily in their learning.

By the simple process of rearranging the curriculum in terms of the pupils' ability, by testing them as to their mental and arithmetical readiness for the new process before trying to teach it to them, by training the pupils to have the proper attitude toward their weaknesses, by developing a spirit of coöperation among the pupils so that the strong ones will aid the weaker ones in working out the well planned remedial program, and lastly, by providing oneself with a well integrated drill program that will increase the pupils' understanding and facility in the use of desirable facts, skills, and information, most of our failures in arithmetic could be avoided.

PRESENT TRENDS AND THEIR SIGNIFICANCE IN THE TEACHING OF MATHEMATICS IN TEXAS[1]

By

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I read somewhere that if 1,000,000 millionaires—men who do not have to look at the right-hand side of the menu—were to order breakfast, 999,999 of them would order ham and eggs. If 1,000,000 teachers of mathematics were to be asked to give the trends, as they see them, in the teaching of this subject, more than likely the replies would not be as uniform as the orders for breakfast. Different sections of the country; the background of the teachers themselves; general environment; opportunities for studying mathematics; the schools in which they have taught; the supervisors, teachers, and class of pupils with whom they have worked; the literature they have read—all of these things and many more will color their replies. Since all men, even members of the Supreme Court, are influenced by community, family, and individual experiences, we neither expect nor desire 999,999 teachers of mathematics to see the same trends in the teaching of mathematics.

Mathematics teachers, however, being accustomed to working with an impersonal attitude, to considering quantitative relationships and logical approaches, and to proving statements based on set assumptions or theorems developed from these assumptions, should be able not only to analyze our various trends, but to render invaluable

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[1] Read before the Mathematics Sections of the North Texas Division of the Texas State Teachers Association at Fort Worth, Texas, March 7, 1936, and of the Sixth Annual Teacher-Training Conference of North Texas State Teachers College at Denton, Texas, March 14, 1936.

service in directing these trends. Here we have an acid test for the transfer of training.

While I count the trends, as I see them, giving references and my own views, you, in the light of your experiences, will weigh, accept, reject, revise, add to what I say and be benefited or not benefited as a result of your own thinking. Study with an open mind all available data, but final decisions, to be reliable for you, must be your own decisions.

The order in which I count is not significant, but the number—13—is. I want to emphasize my Rule No. 13 as much as Dwight Morrow emphasized his Rule No. 6. Also, there are 78 lines connecting each one of the 13 trends I shall count with each other one.

First: Requiring blind memory work is decreasing. In algebra, there was a time when we memorized rules, parrot fashion, and using these rules, still parrot fashion, we performed every exercise on the page below the rule, on the next page, and the next page, on and on until we came to another rule. Another rule for the next group of exercises! The boys and girls who could say the rules in the exact words of the book and who could perform the exercises most rapidly, without pausing to think, of course knew the most algebra. Factoring, rows upon rows of it performed without rhyme or reason, was blind memorization to the n^{th} degree. Wentworth's Algebra used the "substituting z" method, requiring seven or eight lines, to factor the trinomial $ax^2 + bx + c$. If, in the neighborhood of the second or third line, the pupil recognized the final factors, he was lazy, not alert. Some of our new algebra texts dwell on perfect square trinomials and practice for awhile with $a = 1$—a wasteful practice, I think—but the general form, $ax^2 + bx + c$, is always factored mentally even if not too thoughtfully. Last month I said to an Algebra 2 class, after I had asked a pupil to write some small numeral on the board and factor it, that I did not like the definition for factoring given in our text and, as we learned more

about the process, perhaps we could make a definition of our own. We took five or ten minutes each day for a week to discuss our definitions. I assured them that we did not have to adopt any one. Frequently, I heard: "I believe I'll take yours. It sounds better than mine." Copyrights were not considered! One girl said she didn't know before that a definition had any fun to it. However, I was most proud of a quiet boy who said he had been examining other definitions given by the author and added: "He doesn't even attempt to define coefficient." Incidentally, the boy was using the index. While we, as a class, were manifesting interest in his discovery, another boy asked: "Well, how can we define coefficient?" I said some things could be used more easily than they could be defined. I put my handkerchief to my face and said: "I can use my handkerchief but how shall I define it?" We stopped to laugh over a few suggested definitions—I borrowed the thought from David Eugene Smith, who ridiculed defining an elephant before allowing the children to see one. Then I suggested that since we had learned factoring we could formulate this definition: Any factor or factors in an expression may be called the coefficient of the other factor or factors in that expression. During the week I guided them into learning three definite uses for factoring: to change the subject of a formula—correlating with science, if you please; to decrease the number of multiplications—correlating with common sense; and to work with fractions—correlating with the next unit. We put about one-seventh the time on factoring that we did ten years ago, memorized less than one-seventh as much, and, I hope, memorized blindly even less than that.

Second: Crowding the work is decreasing. Instead of crowding in all the mechanical manipulations in the text, we are taking time to think through a few fundamental processes. Such a practice, if followed consistently, beginning with the kindergarten, is its own reward. If, in adding $5/6 - 3/8 - 1/12$, we think $\frac{5}{2.3} - \frac{3}{2.2.2} - \frac{1}{3.2.2}$,

and $\frac{9}{2.3.2.2}$, etc., the L.C.D. and the new numerators have some meaning. Later, should –1 12 become $-\frac{x-2}{12}$, the $-2(x-2)$ will follow naturally from the law of multiplication of negative terms. Similarly, x 2 — 1 3 will not be confused with $\frac{x}{2}$ —1 3. If too much drill work dulls thinking and if we are introducing too many new elements at once, perhaps a few *new* views of important old elements, instead of so many *re*views and half views of too many elements, will be fruitful.

Dr. Alonzo Myers, an educator of national prominence, speaking in Dallas a few weeks ago, advocated no subject matter for the elementary grades. Instead of subject matter he favors activities which teach coöperative living, good morals, good citizenship, and orderly habits of living and thinking. If he can make such recommendations without batting an eye, surely we need not crowd our class hour—I do not imagine that Mark Hopkins and his end-of-log pupil watched a clock. Last week when my pupils in Geometry 2 were interested in constructing a required triangle in several different ways I, a slave of tradition or of a plan book or of both, rushed on to an assigned construction. Lasting results are not obtained by crowding the work for immature children. It reminds me of sewing the length of my doll's quilt without knowing that the bobbin was empty!

Third: Some *why* instead of all *how* is a favorable trend in the teaching of mathematics. Clear lines connect this with Nos. 1 and 2. In $x - (y-z)$, new texts do not mention "minus before parenthesis changes the sign." Parentheses are symbols to show that what is inclosed is to be treated as one term. Hence, $x - (y-z)$ means subtract $(y-z)$ from x. Since subtraction is adding the remainder to the subtrahend to obtain the minuend and multiplication is a short way to add, we may think — 1 times $(y-z)$. Other

illustrations are: Given $2x = 10$, $\frac{2x}{2} = 10/2$; $prt = i$, $\frac{prt}{pr} = \frac{i}{pr}$; easy division, not misleading cancellation. $x + 2 = 7$, $x = 5$, easy subtraction, not magic transposition. Translating $x + 2 = 7$ in several different ways, making formulas, assigning reasonable values to the letters, and talking of the advantages and disadvantages of making up exercises are *why* practices more valuable than countless *hows*.

Fourth: Informal work is increasing. In geometry more intuitive work, more easy exercises, fewer formal proofs, and more informal discussions are recognitions of children's undeveloped span of attention. Instead of long *memorized* proofs for "the square on the side opposite an acute angle, etc.," we talk of the reasonableness of $c^2 = a^2 + b^2$ minus something if the square on the side opposite a right angle is exactly $a^2 + b^2$. Similarly, for obtuse angles. If the *something* is not learned until law-of-cosine days, who cares?

In this connection read "Improving Study Habits," *School Review*, December, 1934; Nyberg's study of the value of interest, *School Science and Mathematics*, November, 1934; and the dozens of articles in *The Mathematics Teacher* during the last two years pertaining to habits of associating ideas, reasonable approximations, intelligent approaches, quantitative relationships, etc. Schorling and Clark's *Units in Mathematics*, A, B, C, D are off the press and should be mentioned here.

Mathematics to be interesting to children must be easy enough for them to learn. Kilpatrick emphasizes children's "liking what they do" instead of "doing what they like"—they can't like formal work which is too difficult for them to do. All of us like what we can do well. I venture the guess if you like tennis that you play well. I can think of only one thing which people persist in doing no matter how poorly they perform, and that is driving an automobile.

Fifth: The growth of general mathematics is noteworthy. Last fall the State of Texas took a long step in the general mathematics direction by adopting free textbooks in the subject. Mr. L. C. McDonald of Houston, writing for *The Texas Outlook*, August, 1935, objects to a Textbook Law, the administration of which determines what is to be studied in Texas; but, since general mathematics has been advocated by leading mathematicians for almost a half century—at least since the Committee of Ten in 1894—Texas is not exercising *too much* initiative in adopting a textbook in that field. E. H. Moore's presidential address delivered before the American Mathematical Society in 1902, reprinted in the first yearbook of the National Council of Teachers of Mathematics is itself sufficient authority for an adoption in Texas—thirty-three years later!

I have a letter from the State Department of Education saying that their newer bulletins will likely recommend general mathematics for the eighth grade, first-year algebra for the ninth grade, and plane geometry for the tenth—two of these three years to be required. Since most of the colleges require only two years of unassigned mathematics for entrance, there is no reason why one of these years may not be general mathematics. Texas Christian University, Fort Worth, not only gives high school general mathematics a strong indorsement, but is enthusiastic over its college courses in general mathematics. Also, North Texas State Teachers College, Denton, is finding experimental college courses in general mathematics highly satisfactory. If colleges will accept general mathematics for entrance credit, I favor requiring at least one year of general mathematics for all high school pupils. Various studies (see *The Mathematics Teacher*, December, 1934) show that pupils who have had these survey courses do better work in college mathematics, and certainly they give boys and girls who are not going to college a broader and more appreciative view of the uses of mathematics.

Cities with a complete Junior High School program can easily introduce general mathematics. For other Texas cities the following plan has the support of mathematical leaders all over the United States, notably Dr. Ralph Beatley, whom all readers of *The Mathematics Teacher* know as chairman of the geometry committee: general mathematics for the eighth grade; beginning algebra—not half-baked algebra tacked onto a general mathematics foundation—for the ninth grade; plane geometry for the tenth grade; and independent half-year courses for the eleventh grade. These half-year courses may be solid geometry, trigonometry, advanced arithmetic, and advanced algebra. Pupils who like mathematics or who can arrange a schedule may elect any or all of these four. With general mathematics and with more mature minds for beginning algebra and geometry, advanced arithmetic will not be as necessary as it has been. Because all colleges offer trigonometry and most of them do not offer solid geometry, I should elect solid geometry in high school instead of trigonometry if I could not get both. Tyler is not offering trigonometry next year for that very reason. The essentials of trigonometry as a foundation for college trigonometry can be taught in general mathematics and in advanced algebra. Mr. Shea, director of mathematics, San Antonio, thinks the essentials of solid geometry are in other courses, and advocates omitting solid geometry. That depends upon what one considers essentials. Solid geometry was essential to me as a foundation for spherical, descriptive, and projective geometry. It is the most delightful course I teach, because it is the only one not elected by Thorndike's much-talked-of less-than-110-I-Q group. We do pay too much attention to building high school courses for the pupils who are going to college when less than 25 per cent are going, but we also pay too little attention to this less than 25 per cent group who *are* going to college and who are our potential leaders. Neglected as they are, do let them keep solid geometry. A definite amount of pure mathematics should be retained in high

school and in college, not only because of the mental satisfaction it affords the students who enjoy it, but because of its possibilities in advancing science—oftentimes, unforeseen possibilities. For instance, radio depends upon formulas which mathematicians had previously developed—not for unforeseen radio but for fun.

Require eighth grade general mathematics for everybody. Require only the one year of general mathematics for the non-college group, but require also ninth grade algebra and tenth grade geometry for the college group. The eleventh grade will not be neglected any more than it is now. With ninth grade pupils more mature for beginning algebra, tenth grade pupils more mature for geometry, better textbooks, less formal work in both subjects, and less poorly prepared teachers, mathematics will grow in popularity until this or that requirement will cease to be a problem. Notice that this proposed schedule has one and one-half years of algebra instead of our present two years; yet with notions of relationships stressed in general mathematics and with one year in the ninth grade where the pupils are more mature, there is no reason why the pupils should not be better prepared for eleventh grade algebra and for college algebra than they are now.

Teachers who are not general-mathematics minded should consider the rapid trend towards no required mathematics. Teaching general mathematics, even in the making, is more desirable than working in another field. However, mathematics teachers who are given non-mathematical classes need not apologize—majors in other subjects have held mathematics classes until we owe them a great many return *holding* visits!

Sixth: The mental age of the child is being considered.

Pupils who are mentally slow often fail in algebra two or three times. Finally, coming to the easy introduction in plane geometry, they are delighted to find something they can do. We have to face them when they come to us, beaming, and say: "I had a hard time with arithmetic and

algebra, but I like this." I have a great contempt for myself when I, helping them to enjoy their first meaningful mathematical experience, praise the pictures they have cut from newspapers or magazines portraying the multitudinous uses of geometry while I know that in three more weeks they cannot select the included angle between the two sides indicated in the exercise or will use "corresponding parts of congruent triangles are equal" before proving the triangles are congruent. Children of this type can and should be helped to learn, by actual experiments, many interesting things *about* mathematics. Certainly general mathematics—acknowledging one of my seventy-eight connecting lines—can be planned to benefit them longer than three weeks.

Another group of pupils appears slow either because they have had slow teachers, mathematically speaking, for their foundation courses or because the pupils themselves are immature. A year or two of general mathematics may smooth the foundation of the former and certainly it will give the latter some much-needed time to mature.

One of my friends says she could not learn algebra because she was told to add, yet sometimes she was supposed to add and sometimes she was supposed to subtract. No one showed her the meaning of the algebraic scale or that the minus symbol could denote direction—an example of poor teaching. Frequently, pupils who fail or barely pass Algebra 1 after three or four years do excellent work in Algebra 4—an example of immaturity. Think of the mental suffering of these three types: the slow, the immature, and the poorly taught. Each of us, in one way or another, has experienced that *insecure* feeling—I am thinking of a banquet where six or eight pieces of silver looked alike. Small wonder that children are indifferent. Indifference is their only secure armor against this dreaded *insecure* feeling. Nature provides all animals with self-protection instincts!

At least one of the editorial writers of *The Dallas Morning News* should be a member of our curriculum revision

committee. Last week he wrote: "True democratic education is not giving the same education to all, but giving all the opportunity to show capacity. Once inherent capacity is made manifest, each should be able to get the sort of education needed for suitable mental development." In a well-ordered society there should be a place for minds of low, medium, and high capacity—the degree of capacity being measured by the *way* a person does his work and not by the *kind* of work he does. A well-ordered society should not allow those to be born for whom there is to be no place.

Seventh: A tendency to discuss, if not to practice, what I, for want of a better name, shall call new mathematics. New mathematics is any one or any combination of the following: general, fusion, correlation, integration, unified, experimental (see Dr. Ettlinger's article in the February *Texas Outlook)*, paralleling courses, and, quoting Charles Russel Atherton of Columbia University (see *The Mathematics Teacher,* January, 1935), "courses involving mathematics which slide over any reference to mathematics." Refer to any bibliography of recent mathematical productions—I recommend "Mathematical Training for Economic Thinking and Social Mindedness" by William L. Schaaf, *The Mathematics Teacher,* December, 1934, and also Dr. Schaaf's book outlining mathematical topics which should be included in a modern course of study—turn at random to any three, and you will find two of these three discussing some phase of new mathematics. Mathematics not recognized as mathematics is a deplorable phase. Of course mathematics will contine as long as civilization continues, but why discard a time-honored name? As Dr. Hedrick says, the general public looks through the newspapers or surveys business practices, sees no x's, and calmly announces: No algebra here. Algebra is not x but a symbolism for thinking about numbers, and includes any intelligent discussion of relations between measures of quantities.

Two weeks ago at the Dallas institute one of our speakers described in detail one unit of integration as practiced in Lincoln School, New York. As an afterthought, she said: "As yet foreign languages and mathematics have no part in this experiment." As she talked, I had made note of five distinct uses of mathematics in the unit she described; while Latin, self-effacing friend that she is, had supplied the name itself—integration. I am not talking for or against integration. The point I want to make is that when mathematics is used, mathematics should be recognized. "Render unto Caesar the things which are Caesar's."

Eighth: Less mathematics is being required for high school graduation and for college entrance, and fewer pupils are electing the non-required courses. A table in *The Mathematics Teacher*, October, 1933, shows this trend in general, and I have replies to letters, written since I began the study of this topic, which show the trend in Texas. The State Department of Education notes a decrease in the number of pupils who are electing mathematics; so do Dallas, San Antonio, Fort Worth, and Houston. Austin High School, since 1930, has decreased the mathematical requirement for graduation from three units to two units—a significant decrease in keeping with the decrease in the entrance requirement of The University of Texas. No high school mathematics is required in San Antonio or in Fort Worth, and the requirement has been decreased in Houston since 1930. (Houston did not give me the figures.) Most colleges require less mathematics for entrance than they did five or ten years ago. By the year 2000, if the trend continues, practically no mathematics will be required for college entrance. In 1800 Harvard required no mathematics for entrance and Yale and Columbia required only arithmetic. By 1900 most colleges were requiring at least three years of high-school mathematics for entrance. By 1925 the pendulum was swinging back toward 1800. While, generally speaking, secondary schools "goose step" along the college-entrance-requirement

curve, other influences decreasing the study of mathematics in high school are: less maturity and less ability among the pupils attending high school; a high percentage of failures in high school mathematics—though Dr. Nathan Altshiller-Court, Vice-President of the Mathematical Association of America, declares that high school pupils do not fail in mathematics because they know less about it than they know about the subjects they pass but because mathematical measuring rods cannot be ignored as can the measuring rods in other subjects—and the pushing of secondary mathematics down into the junior high school grades. The last-named influence may be a blessing in disguise, because while we are floundering around looking for something which pupils of junior high school age can learn, we are discovering the limitless possibilities of easy mathematics.

Ninth: The contribution of improved textbooks.

One criticism of secondary schools has been that pupils, not being taught to follow printed directions, are helpless without a teacher. Perhaps we, who enjoy talking, have neglected the textbook explanations, but, in all fairness, poor or no textbook explanations made some of our talking a necessary evil. With the new self-teaching texts, instructors and pupils are learning to study.

There is barely time to name: *Tenth*, Units, a unifying influence, as the name implies—I wish I could tell you a few amusing misinterpretations of this trend; *Eleventh*, Objectives for teachers and pupils with their manifold outcomes; and *Twelfth*, Junior high school influences, which deserve more than a passing mention.

Thirteenth: Thirteenth, the poor teaching of mathematics, is not a trend but a standstill. Dr. Reeve says that mathematics at its best needs no defense. Mathematics cannot be at its best until teaching is at *its* best. Dr. Breslich (*The Mathematics Teacher*, October, 1933) gives as one of his four reasons for slow progress in mathematics the lack of preparation of teachers to put new

ideas into practice. Even states ranking high educationally require fewer semester hours in mathematics for mathematics teachers than is required of other teachers in the subjects which they teach. Texas requires zero semester hours. Not one-third of the teachers of mathematics in Texas have majored in mathematics. I do not want to overemphasize teaching one's major. When I talked to the North Texas Teacher-Training Conference two years ago on the "Characteristics of a Progressive Mathematics Teacher," I weighed mathematical scholarship as ½ of 33⅓ per cent, teacher-training as ½ of 33⅓ per cent, and capacity for growth as 2/2 of 33⅓ per cent. One can grow into an excellent teacher. Incidentally, increasingly heavy teaching loads play havoc with scholarship, capacity for growth, and any other 33⅓ per cent. Also, there are individuals with mathematical scholarship so wholly lacking in the other qualities of a progressive teacher that they are barely ½ of 33⅓ per cent excellent. Perhaps they are physically lazy, too indifferent or too conservative to adjust themselves to new ideas, too prone to glory in making mathematics difficult, too selfish to work for the future of mathematics, etc., etc. A straw indicating our lack of professional spirit is our 122 Texas members of the National Council of Teachers of Mathematics. Dallas alone has 79 white high school mathematics teachers, yet all of Texas, including elementary, high school, and college teachers, has a bare 122 who affiliate themselves with their own organization.

With eleven positive trends in the teaching of mathematics in Texas: (1) Blind memory work is decreasing; (2) Crowding the time is decreasing; (3) More *why* instead of all *how* is being considered; (4) Informal approaches are increasing; (5) The growth of general mathematics is noteworthy; (6) Mental age is being considered; (7) New Mathematics is being weighed; (9) Better textbooks; (10) Units; (11) Objectives; (12) Junior high

school—with eleven positive trends, we, all working together, can rid No. 13, poor teaching of mathematics, of proverbial No. 13 significance, and then No. 8, less mathematics being required and less of the non-required being elected, will automatically right-about-face.

THE BROWN UNIVERSITY PRIZE EXAMINATION

The Brown University Prize Examination for freshmen was given on Saturday, October 10, 1936. The three prizes were awarded as follows:

First: Evans Munroe, of Houston.

Second: William S. Swayze, of Kelly Field.

Third: William A. Purdy, of Sulphur Springs.

A good paper was also written by Kenneth C. Clark, of Austin.

The questions were as follows:

1. Determine k so that $x + 3$ will divide $x^3 + kx^2 - 2x - k^2$ with a remainder equal to -10.

2. At his usual rate a man can row 15 miles downstream in 5 hours less time than it takes him to return. If he could double his rate, his time downstream would be only one hour less than his time upstream. Find his usual rate in still water and the rate of the current.

3. Inscribe a square in a semicircle of radius R. Compute the length of a side of the square.

4. Prove that the sum of the squares of the segments of two perpendicular chords of a circle is equal to the square of the diameter.

SOME PUZZLES

After an exact long division had been completed, one particular digit was replaced by the letter c wherever it occurred. Then all the other digits indiscriminately were each replaced by the letter x. Here is the result:

$$
\begin{array}{r}
x\,c\,x\;)\;x\,x\,x\,x\,c\,x\,x\;(\;x\,x\,c\,x \\
x\,x\,c\,c \\
\hline
x\,x\,x\,c \\
x\,x\,c \\
\hline
x\,x\,x\,x \\
x\,c\,x\,x \\
\hline
x\,x\,x\,x \\
x\,x\,x\,x \\
\hline
\end{array}
$$

Reconstruct all the original figures. (From *The American Mathematical Monthly* for May, 1936, page 305.)

The cube root of an eight-digit number was extracted in the usual manner, and then each digit was replaced by a code letter, with the following result. Solve the code.

W P, W D B, M W C *B P*	*A K C*
A S W D B *B P K P B*	*A M K M*
B C M S M W C *B C M S M W C*	*M M S S P C*

(From *The American Mathematical Monthly* for June–July, 1936, page 374.)

In a certain bank there were eleven distinct positions; namely, in decreasing rank, President, First Vice-President, Second Vice-President, Third Vice-President, Cashier, Teller, Assistant Teller, Bookkeeper, First Stenographer, Second Stenographer, and Janitor. These eleven positions are occupied by the following, here listed alphabetically, Mr. Adams, Mrs. Brown, Mr. Camp, Miss Dale, Mr. Evans, Mrs. Ford, Mr. Grant, Miss Hill, Mr. Jones, Mrs. Kane, Mr. Long. Concerning them the following facts only are known:

1. The Third Vice-President is the pampered grandson of the President, but is disliked by both Mrs. Brown and the Assistant Teller.
2. The Assistant Teller and the Second Stenographer shared equally in their father's estate.
3. The Second Vice-President and the Assistant Teller wear the same style of hats.
4. Mr. Grant told Miss Hill to send him a stenographer at once.
5. The President's nearest neighbors are Mrs. Kane, Mr. Grant and Mr. Long.

6. The First Vice-President and the Cashier live at the exclusive Bachelors' Club.

7. The janitor has occupied the same garret room since boyhood.

8. Mr. Adams and the Second Stenographer are leaders in the social life of the younger married set.

9. The Second Vice-President and the Bookkeeper were once engaged to be married to each other.

10. The fashionable Teller is son-in-law of the First Stenographer.

11. Mr. Jones regularly gives Mr. Evans his discarded clothing to wear, without the elderly Bookkeeper knowing about it.

Show how to match correctly the eleven names against the eleven positions occupied. (From *The American Mathematical Monthly* for August–September, 1936, page 432.)

A mule and a donkey were walking along, laden with corn. The mule says to the donkey: "If you gave me one measure, I should carry twice as much as you. If I gave you one, we should both carry equal burdens." Tell me their burdens, O most learned master of geometry.—A riddle attributed to Euclid. From "Palatine Anthology," 300 A.D. (From *Mathematical Wrinkles,* by S. I. Jones, page 92.)

Democharés has lived a fourth of his life as a boy; a fifth as a youth; a third as a man; and has spent 13 years in his dotage. How old is he?—From a collection of questions by Metrodorus, 310 A.D. (From *Mathematical Wrinkles,* by S. I. Jones, page 93.)